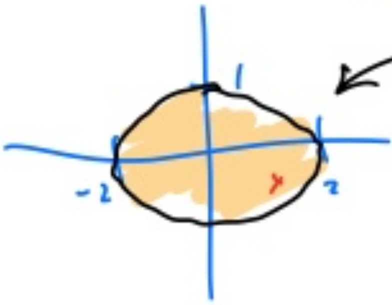
Example

2.44 Find global extrema of $F(x,y) = x^2+3xy+7y^2+7y+8$ on $D = \{(x,y) : x^2+4y^2 \le 4\}$.



closed & bounded $\Rightarrow$ cpt.

By Extreme Value Thm, the cont. fcn. F will attain its global max & min on D. (either in interior or bdry).

Interior: $\nabla F = 0$

$$2x+3y=0 \Rightarrow x = -\frac{3}{2}y$$

$$3x+14y+7=0$$

$$-\frac{9}{2}y + 14y = -7$$

$$\frac{19}{2}y = -7 \Rightarrow y = \frac{-14}{19}$$

$$x = \frac{21}{19}$$

$$\left(\frac{21}{19}, \frac{-14}{19}\right) \rightarrow F(x,y) = 5.421$$

global min.

Boundary: $\nabla f = \lambda \nabla g$ $\quad g(x,y) = x^2+4y^2 = 4$

$$2x+3y = \lambda(2x)$$

$$3x+14y+7 = \lambda(8y)$$

$$x^2+4y^2 = 4$$

Sagemath: $x,\ y,\ f(x,y)$

2 cr pts: $(0.711,\ 0.935,\ 23.16)$ global max

$(1.237,\ -0.786,\ 5.436)$

Maximize $F(x,y,z)$
Restricted to $R =$

To find global max
- $\nabla F = 0$ interior
- each surface $g(x,y,z) = c$

$$\begin{cases} \nabla F = \lambda \nabla g \\ g = c \end{cases}$$

- where surfaces meet — on curves

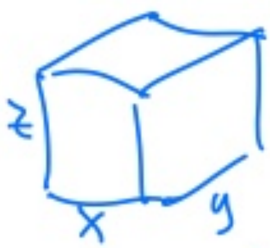
$g_1 = c_1$
$g_2 = c_2$

$$\nabla F = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$
$$g_1 = c_1$$
$$g_2 = c_2$$

- Check all the endpoints.

More examples of optimization

(1) We will construct a box that hold a very corrosive substance. The box holds 2000 cubic feet of this substance.



The bottom material costs \$20/ft^2. The top costs \$25/ft^2. Three of the sides cost \$30/ft^2, and one of the sides costs \$40/ft^2.

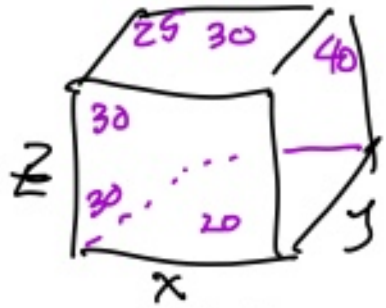
How do we design the box so that the cost is minimum?

$F(x,y,z) =$ cost of box. ← want to minimize this.

constraint $V(x,y,z) = 2000$

$V(x,y,z) = xyz = 2000$

we will use Lagrange.



$$F(x,y,z) = \underbrace{25xy}_{\text{top}} + \underbrace{20xy}_{\text{bottom}} + \underbrace{30(xz+xz+zy) + 40zy}_{\text{sides.}}$$

$$F(x,y,z) = 45xy + 60xz + 70zy$$

Lagrange: $\nabla F = \lambda \nabla V$, $V = C$

$$\begin{cases} 45y + 60z = \lambda yz & \leftarrow \text{mult by } x \\ 45x + 70z = \lambda xz & \leftarrow \text{mult by } y \\ 60x + 70y = \lambda xy & \leftarrow \text{mult by } z \\ xyz = 2000 \end{cases}$$

$$\Rightarrow \begin{cases} 45xy + 60xz = \lambda xyz \\ 45xy + 70yz = \lambda xyz \\ 60xz + 70yz = \lambda xyz \\ xyz = 2000 \end{cases}$$

$$\cancel{45xy} + 60xz = \cancel{45xy} + 70yz$$

$$60xz - 70yz = 0$$

$$\underset{\text{must be} > 0}{z}(60x - 70y) = 0 \Leftrightarrow 60x = 70y \Rightarrow \boxed{x = \frac{7}{6}y}$$

Subtract

$45xy + \cancel{70yz} = 60xz + \cancel{70yz}$

$\Rightarrow \quad 45xy - 60xz = 0$

$x(45y - 60z) = 0$

can't be 0

$45y = 60z$

$z = \frac{45y}{60} = \frac{3y}{4}$

$\boxed{z = \frac{3y}{4}}$

Constraint $\quad xyz = 2000$

$\left(\frac{7}{6}y\right) y \left(\frac{3y}{4}\right) = 2000$

$\frac{7}{8} y^3 = 2000$

$\Rightarrow y^3 = \frac{16000}{7} \Rightarrow y = \sqrt[3]{\frac{16000}{7}}$

$= 13.17\text{ft}$

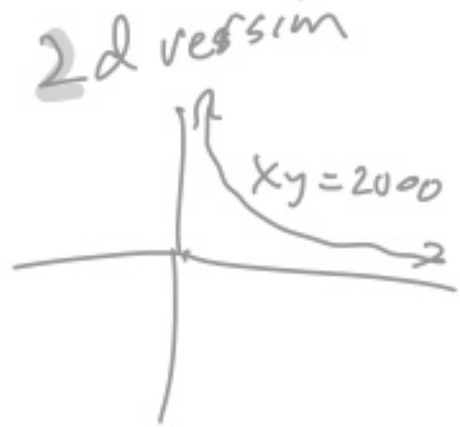
$x = \frac{7}{6}y = \frac{7}{6}\sqrt[3]{\frac{16000}{7}} = 15.37\text{ft.}$

$z = \frac{3}{4}y = \frac{3}{4}\sqrt[3]{\frac{16000}{7}} = 9.880\text{ft.}$

Only 1 cr pt $\quad (x, y, z) = (15.37\text{ft}, 13.17\text{ft}, 9.88\text{ft})$

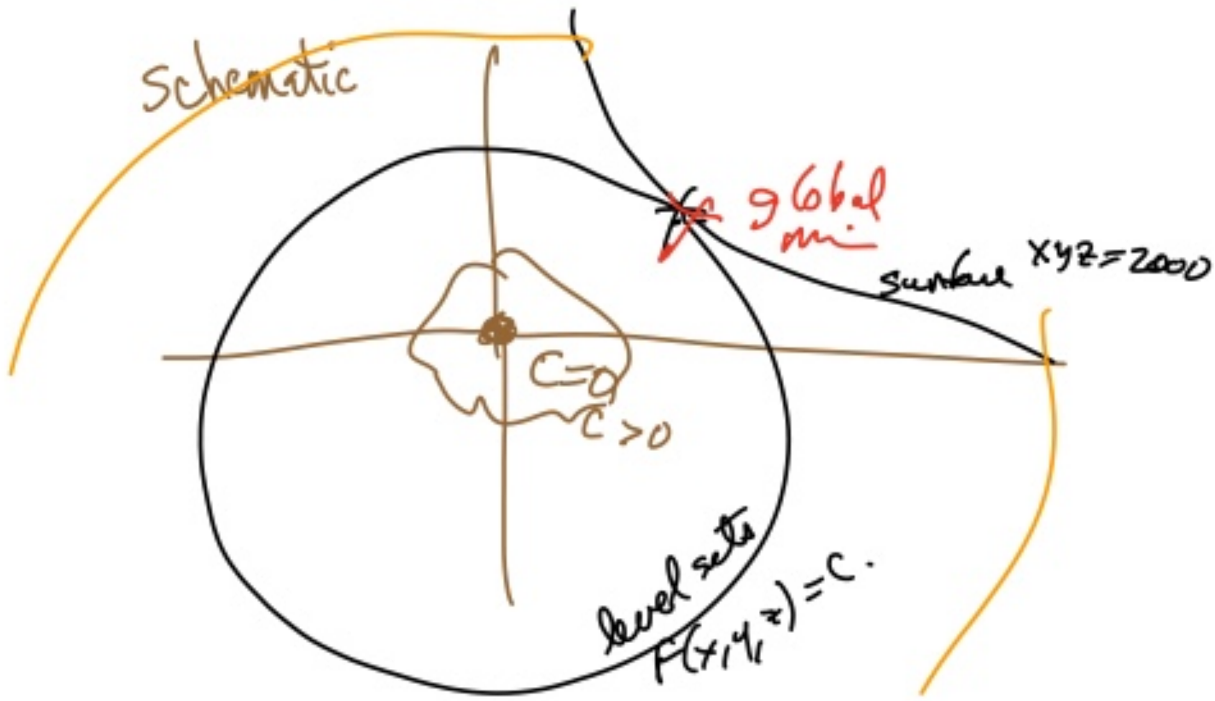
Note: We don't have a max — because we can make the box like this and make the cost $\to \infty$.

Issue $xyz=2000$ is <u>not compact</u> (because not bounded) — so the extreme value thm does not apply.

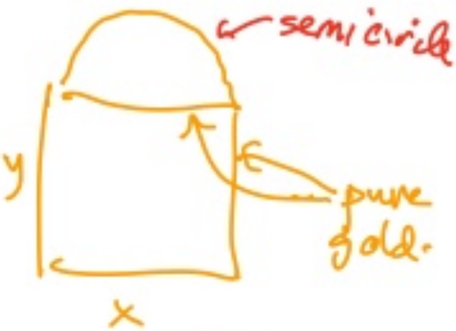


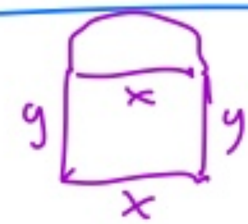
Meanwhile, the level sets of

$$F(x,y,z)=45xy+60xz+70yz=C$$



Example. Our special window should have 30 sq ft of area — in this shape. How do we design it so we use the least amount of gold?





Function: Length of gold. radius

$$F(x,y) = 2x + 2y + \pi\left(\frac{x}{2}\right)$$

Constraint:

$$= \left(2 + \frac{\pi}{2}\right)x + 2y.$$

Area

$$A(x,y) = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$A(x,y) = xy + \frac{\pi}{8}x^2$$

Lagrange: $\nabla F = \lambda \nabla A$

$A = c$

$$\begin{cases} 2 + \frac{\pi}{2} = \lambda\left(y + \frac{\pi}{4}x\right) \\ 2 = \lambda x \Rightarrow \lambda = \frac{2}{x} \quad \text{OK since } x > 0 \\ xy + \frac{\pi}{8}x^2 = 30 \end{cases}$$

$$2 + \frac{\pi}{2} = \frac{2}{x}\left(y + \frac{\pi}{4}x\right)$$

$$\begin{cases} 2 + \frac{\pi}{2} = 2\frac{y}{x} + \frac{\pi}{2} \Rightarrow 2 = 2\frac{y}{x} \Rightarrow \boxed{x = y} \\ xy + \frac{\pi}{8}x^2 = 30 \end{cases}$$

$$x^2 + \frac{\pi}{8}x^2 = 30$$

$$\left(1+\frac{\pi}{8}\right)x^2 = 30$$

$$x^2 = \frac{30}{1+\frac{\pi}{8}}$$

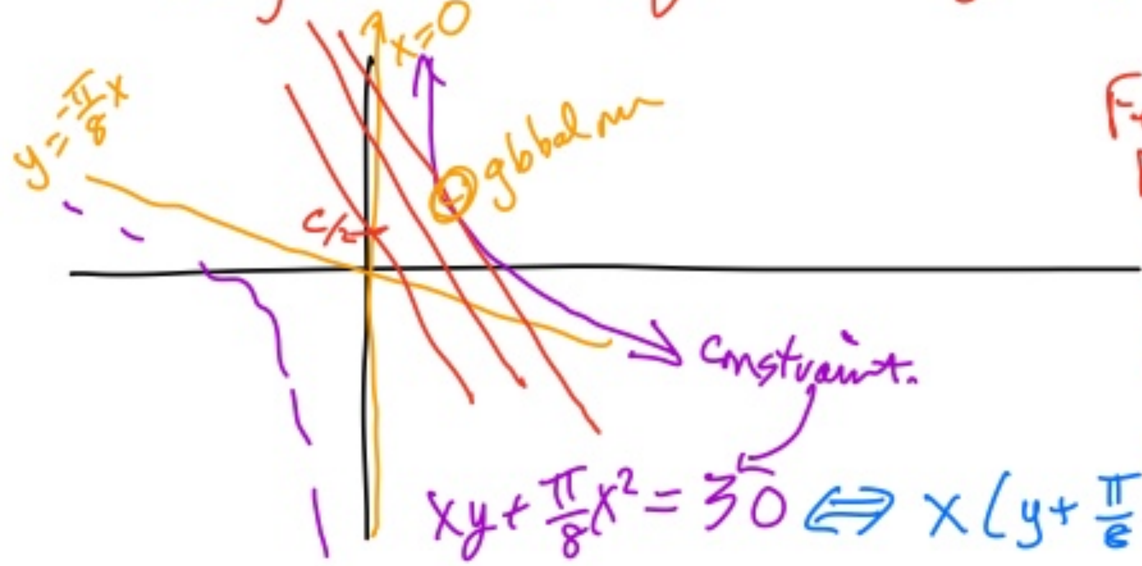
1 cr pt. $\boxed{x = \sqrt{\frac{30}{1+\frac{\pi}{8}}} = y}$

Note: Our calculation would work if we did not specify the area, but just let it be a constant B.

$$\Rightarrow x = \sqrt{\frac{B}{1+\frac{\pi}{8}}} = y$$

Then we would know how to design such a window for any area.

Why is this the global min of length of gold?



Function $F(x,y) = \left(2+\frac{\pi}{2}\right)x + 2y = C$

$$y = -\frac{\left(2+\frac{\pi}{2}\right)}{2}x + \frac{C}{2}$$

$$xy + \frac{\pi}{8}x^2 = 30 \iff x\left(y + \frac{\pi}{8}x\right) = 30$$